Tutored project report No. 1

Ivan Coppo and Valentin Rougier

Université de Nice - Sophia Antipolis

(Dated: May 29, 2012)

The purpose of this report is to explain the steps that led to the different findings made on the considered system — a pendulum.

The system is composed by a simple spherical pendulum, which weight is suspended to a post, being itself fixed on a rotative disc. When the angle between the gravity and the post is non zero, it should have a *chaotic* behaviour. We will aim on studying this system when chaos appears.

We will first consider the most simple case we can encounter with a pendulum, and then gradually increase its complexity.

Contents

I. CASE OF THE SIMPLE PENDULUM

To begin, we will study the case of the simple pendulum. It is the most simple system we can think of when it comes to the pendulums. It is composed of a weight hanging with a inextensible string, which only moves in a plane. The position of the weight will be defined by the length of the string l, and by the angle θ between the string and the vertical axis. To have this configuration with our pendulum, the disc must not rotate.

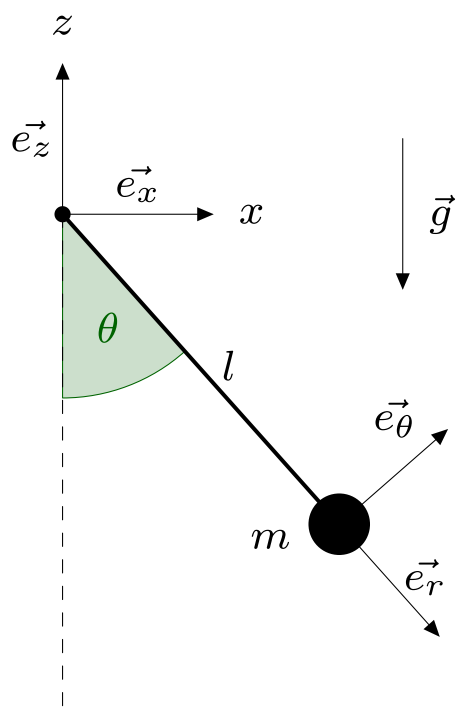


Figure 1: simple pendulum

A. Finding the equations of movement

The first thing to do is to find the weight's coordinates and their derivatives:

$$\begin{cases} x = l\sin\theta \\ y = 0 \\ z = -l\cos\theta \end{cases} \Rightarrow \begin{cases} \dot{x} = l\dot{\theta}\cos\theta \\ \dot{z} = l\dot{\theta}\sin\theta \end{cases}$$

We will now calculate the kinetic and potential energies, and deduce the system's Lagrangian:

$$\begin{aligned} E_k &= \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\right)^2 \\ &= \tfrac{1}{2}ml^2\dot{\theta}^2 \\ E_p &= -\vec{W}\cdot\vec{r} = mgz = -mgl\cos\theta \end{aligned}$$

The Lagrangian is:

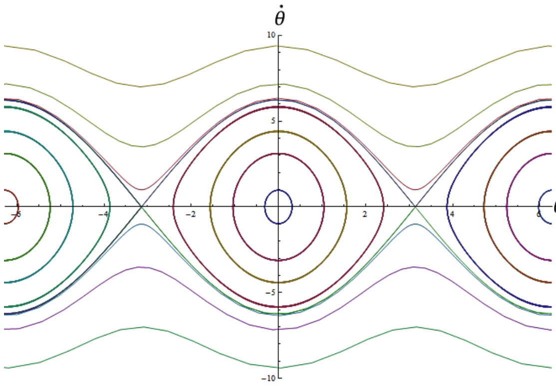
$$\begin{aligned} L &= E_k - E_p \\ &= \tfrac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \end{aligned}$$

The Euler-Lagrange equation gives us:

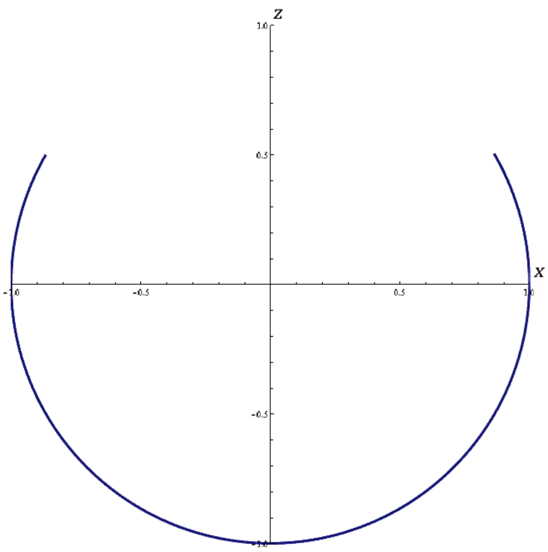
$$\frac{\partial L}{\partial \theta} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} \quad \Rightarrow \quad \ddot{\theta} = -\frac{g}{l}\sin\theta \tag{1}$$

B. Integration of the equation

We want to have the graph of the system's trajectory and its phase portrait. To achieve that, we integrate the equations found above with Wolfram Mathematica. As we could expect, the trajectory is circular.



(a) Phase portrait of the simple pendulum



(b) Trajectory of the simple pendulum

Figure 2: Simple pendulum

II. CASE OF A SPHERICAL PENDULUM

We will then consider the second most simple case that can be encountered with this system – it becomes the equivalent of a simple spherical pendulum. This behaviour occurs when the angle between the post and the gravity – thereafter identified by α – is zero.

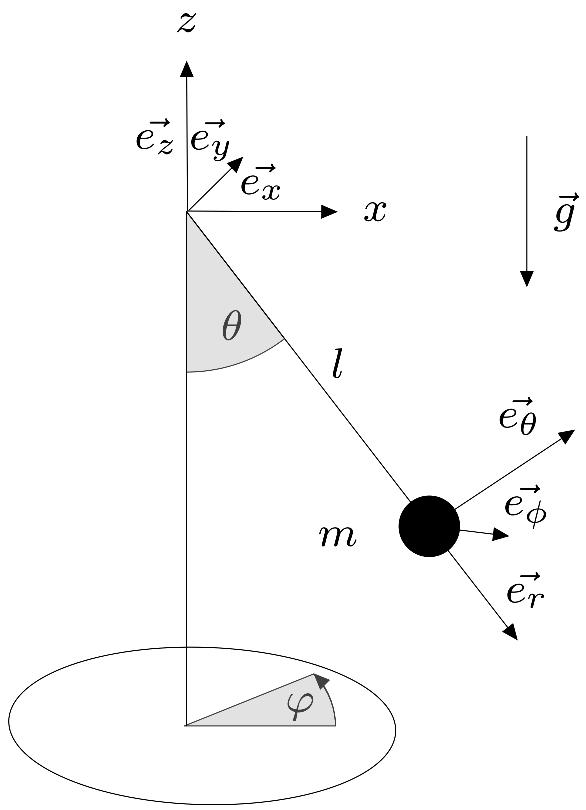


Figure 3: A spherical pendulum

A. Equations of movement

The first thing to do is to find the weight's coordinates and their derivatives:

$$\begin{cases} x = l\sin\theta\cos\varphi \\ y = l\sin\theta\sin\varphi \\ z = -l\cos\theta \end{cases} \Rightarrow \begin{cases} \dot{x} = l(\dot{\theta}\cos\theta\cos\varphi - \dot{\varphi}\sin\theta\sin\varphi) \\ \dot{y} = l(\dot{\theta}\cos\theta\sin\varphi + \dot{\varphi}\sin\theta\cos\varphi) \\ \dot{z} = l\dot{\theta}\sin\theta \end{cases}$$

From this we can easily find the system's kinetic and potential energy, write down its Lagrangian:

$$
\begin{aligned}
E_k &= \frac{1}{2}mv^2 = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{ml^2}{2}(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) \\
E_p &= -\vec{W}\cdot\vec{r} = mgz = -mgl\cos\theta
\end{aligned}
$$

$$
L = E_k - E_p = \frac{ml^2}{2}(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + mgl\cos\theta
$$

We then deduce the Euler-Lagrange equations:

$$
\frac{\partial L}{\partial \theta} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} \Rightarrow \ddot{\theta} = \dot{\varphi}^2 \sin\theta\cos\theta - \frac{g}{l}\sin\theta \tag{2}
$$

$$
\frac{\partial L}{\partial \varphi} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}} \Rightarrow \ddot{\varphi} = -2\dot{\theta}\dot{\varphi}\cot\theta \tag{3}
$$

B. Integrating the equations

We now have a system of two dependant differential equations (1) and (2). We will now integrate them using the computer program Wolfram® Mathematica to obtain the spherical pendulum's phase portrait and trajectories.

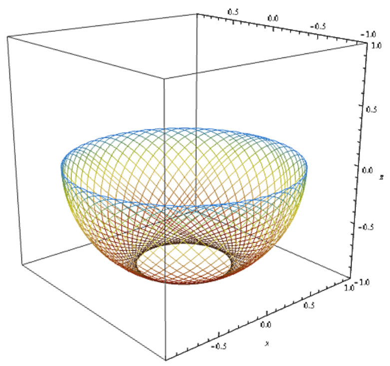


Figure 4: Trajectory of the spherical pendulum

We can notice that the center area will never be reached by the pendulum.

III. CASE OF AN INCLINED SPHERICAL PENDULUM

Now that we know the equations describing a spherical pendulum, we will study the case of the same pendulum, but inclined by an angle α from the z axis in the yOz plan, and without a disc. That is the same as bending the gravity with that same angle. The only difference between this case and the latest will be the expression of the potential energy.

A. Equations of movement

We are going to proceed the same way we did for the spherical pendulum. The coordinates and speed (and thus the kinetic energy) are the same as for the last case. We only have to determine the new expression of the potential energy:

$$\begin{aligned}
E_p &= -\vec{W}\cdot\vec{r} = (mg\cos\alpha\ \vec{e_z} - mg\sin\alpha\ \vec{e_y})\cdot(z\vec{e_z} + y\vec{e_y}) \\
&= (mg\cos\alpha\ \vec{e_z} - mg\sin\alpha\ \vec{e_y})\cdot(-l\cos\theta\ \vec{e_z} + l\sin\theta\sin\varphi\ \vec{e_y}) \\
&= -mgl(\sin\theta\sin\varphi\sin\alpha + \cos\theta\cos\alpha)
\end{aligned}$$

Which gives the new Lagrangian and Euler-Lagrange equations:

$$L = \frac{ml^2}{2}(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + mgl(\sin\theta\sin\varphi\sin\alpha + \cos\theta\cos\alpha)$$

$$\frac{\partial L}{\partial \theta} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} \Rightarrow \ddot{\theta} = \dot{\varphi}^2\sin\theta\cos\theta + \frac{g}{l}(\cos\theta\sin\varphi\sin\alpha - \sin\theta\cos\alpha) \tag{4}$$

$$\frac{\partial L}{\partial \varphi} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}} \Rightarrow \ddot{\varphi} = \frac{g}{l\sin\theta}\cos\varphi\sin\alpha - 2\dot{\theta}\dot{\varphi}\cot\theta \tag{5}$$

B. Integrating the equations

In the same way as we did before, we integrate the newly found equations to get the pendulum's trajectory. We find that, as it could be expected, this case is the same as the spherical pendulum, except that its trajectory will be bended: the minimal of the gravitational potential has shifted by an angle α, and the pendulum will rotate around it.

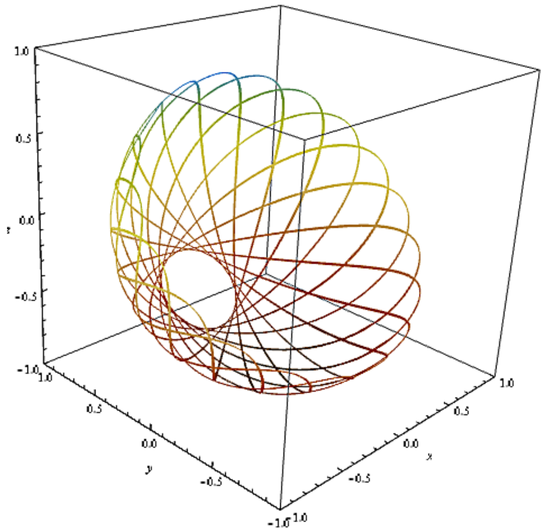


Figure 5: Trajectory of an inclined spherical pendulum

In this case, we still don't observe chaos.

IV. CASE OF THE CHAOTIC PENDULUM

To observe chaos with our pendulum, the disc has to have a mass M$\neq$ 0 and a radius R$\neq$ 0. We also want α to be non-zero.

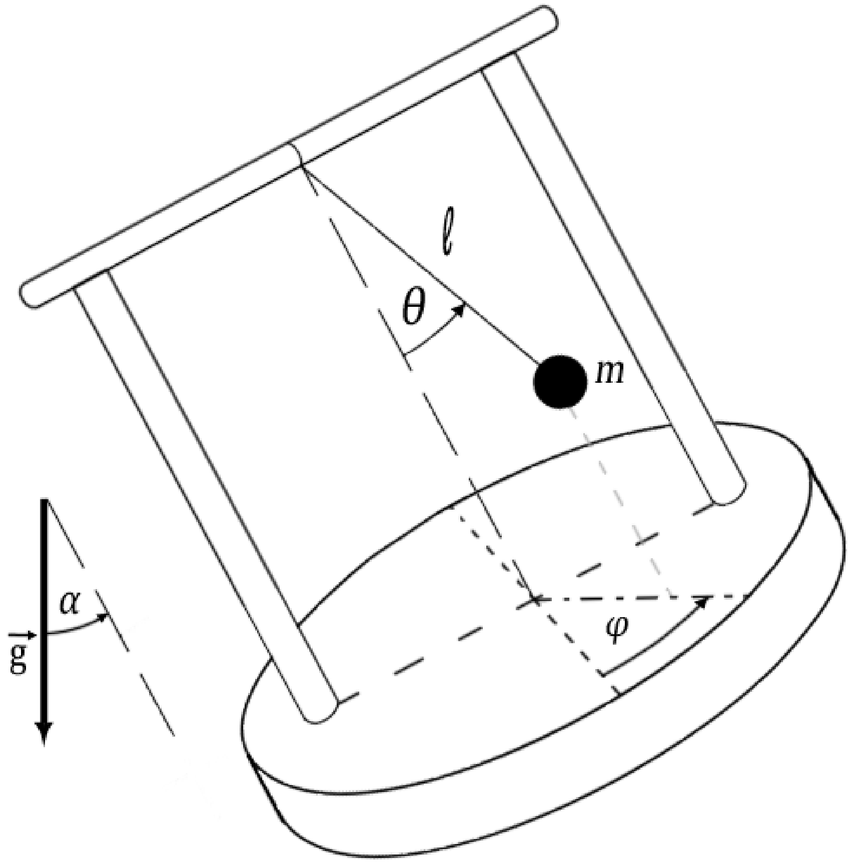


Figure 6: the chaotic pendulum

A. Equations of movement

The only noticeable difference between this case and the case of the inclined spherical pendulum will be the kinetic energy's expression:

$$E_k = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + \frac{1}{2}MR^2\dot{\varphi}^2$$

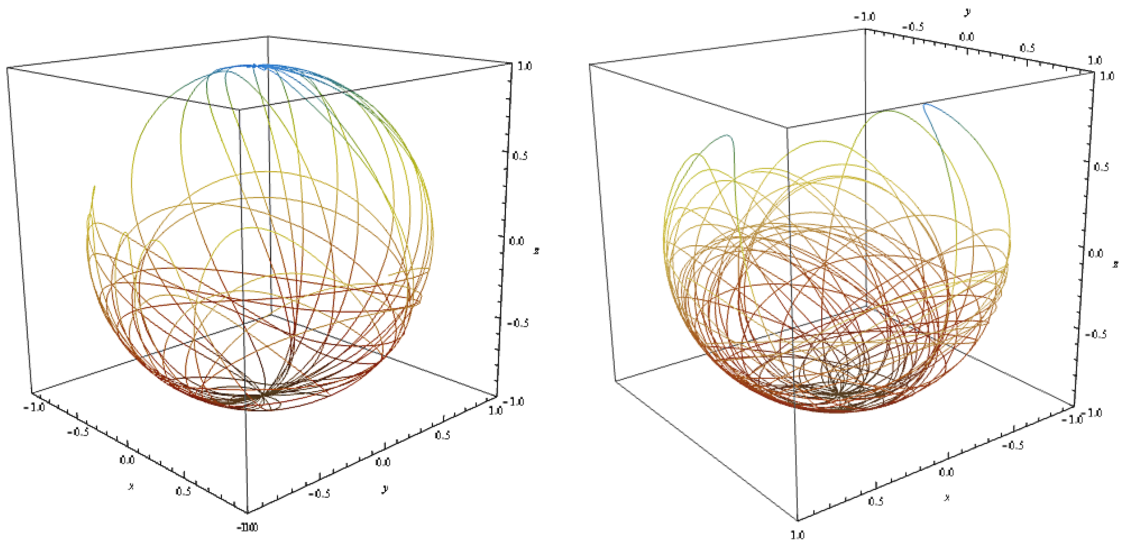
$$L = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + \frac{1}{2}MR^2\dot{\varphi}^2 + mgl(\sin\theta\sin\varphi\sin\alpha + \cos\theta\cos\alpha)$$

$$\frac{\partial L}{\partial \theta} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} \Rightarrow \ddot{\theta} = \dot{\varphi}^2 \sin\theta\cos\theta + \frac{g}{l}(\cos\theta\sin\varphi\sin\alpha - \sin\theta\cos\alpha) \tag{6}$$

$$\frac{\partial L}{\partial \varphi} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}} \Rightarrow \ddot{\varphi} = \frac{ml\sin\theta(g\cos\varphi\sin\alpha - 2l\dot{\theta}\dot{\varphi}\cos\theta)}{ml^2\sin^2\theta + MR^2} \tag{7}$$

B. Integrating the equations

As we did previously, we now integrate the pendulum's equations to get its trajectory, depending on the initial conditions.



(a) A trajectory

(b) A trajectory for different initial conditions

Figure 7: Trajectory of the chaotic pendulum

To study more deeply the case of the chaotic pendulum, we will need to use a new notion : the Poincaré section.

V. POINCARÉ SECTION

A. General case

In order to characterize our system's behaviour, we will need to use a Poincaré section. In our case, a Poincaré section can be seen as a plan in space, on which we will place the points where the pendulum passes.

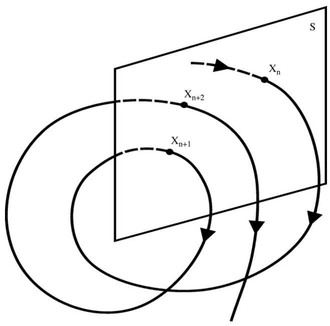


Figure 8: Principle of the Poincaré section

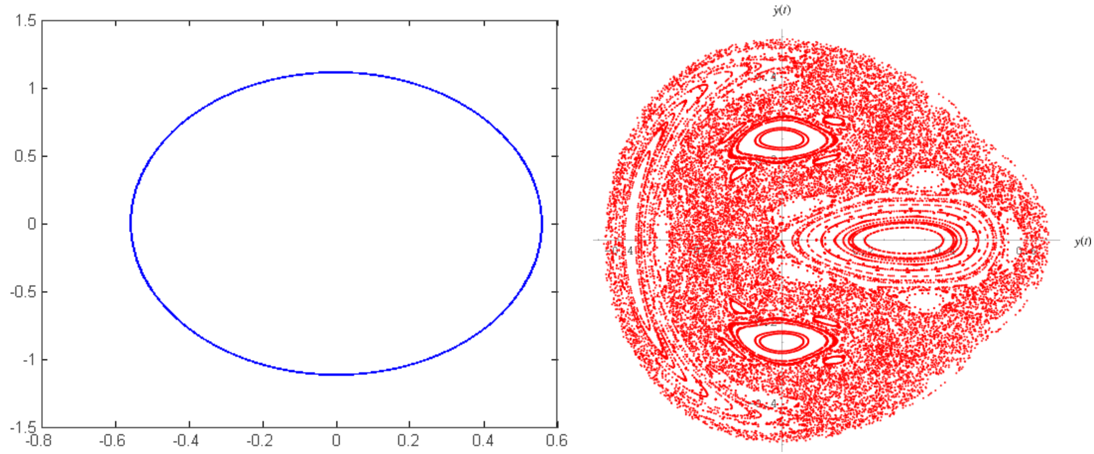
The Poincaré section's graph can be :

a single point: the system is periodic ;

a discrete set of points: the system is periodic too ;

a closed curve: the system is quasi-periodic ;

a cloud of points: the system is chaotic.



(a) Quasi-periodic system - image credits : LASIN, uni-lj [5]

(b) chaotic system (image credits : Nelson De Leon, Indiana University)

Figure 9: Two examples of Poincaré sections

We now want to use the Poincaré section to know the pendulum's behaviour (periodic, quasi-periodic, chaotic). We will study the four different cases that we described in the parts I, II, III and IV.

B. Poincaré section of the chaotic pendulum

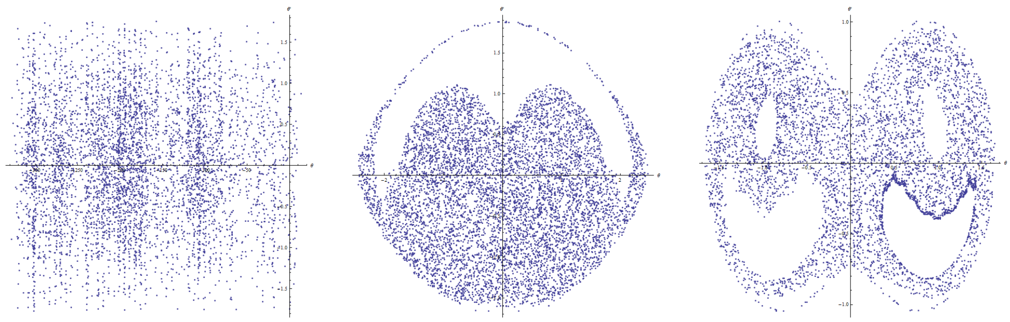


Figure 10: Some examples of the chaotic pendulum's Poincaré section

VI. STABILITY ANALYSIS

We will now make a stability analysis

A. Nondimensionalization

Since the equations (6) and (7) are quite complex, we will want to make them simpler. We will nondimentialize them to only have one or two control parameters left, as there are 6 up to now.

We can first recognise that $\frac{g}{l}$ is equivalent to a squared frequency, which we will call ω. Thus we can write:

$$\sqrt{\frac{g}{l}} = \omega$$

With that in mind, we can write:

$$\begin{aligned}
\ddot{\theta} &= \dot{\varphi}^2 \sin\theta\cos\theta + \omega^2(\cos\theta\sin\varphi\sin\alpha - \sin\theta\cos\alpha) \\
\ddot{\varphi} &= \frac{\sin\theta(\omega^2\cos\varphi\sin\alpha - 2\dot{\theta}\dot{\varphi}\cos\theta)}{\sin^2\theta + \frac{MR^2}{ml^2}}
\end{aligned}$$

As we did for $\frac{g}{l}$, we will consider that $\beta = \frac{MR^2}{ml^2}$ is a control parameter. It represents the report between the disc's and the pendulum's moments of inertia.

Finally we will also rescale the time. To achieve that, we only need to take $\tilde{t} = \frac{t}{\tau}$, where $\tau = \frac{1}{\omega}$ is our system's characteristic time. The equations become:

$$\ddot{\theta} = \dot{\varphi}^2 \sin\theta\cos\theta + \cos\theta\sin\varphi\sin\alpha - \sin\theta\cos\alpha \tag{8}$$

$$\ddot{\varphi} = \frac{\sin\theta(\cos\varphi\sin\alpha - 2\dot{\theta}\dot{\varphi}\cos\theta)}{\sin^2\theta + \beta} \tag{9}$$

B. Study of the fixpoints

We first need to find the different fixed points of our system. A fixed point is a point with initial conditions such that the system does not evolve after any period of time. There are stable and unstable fixed points. To find fixpoints, we must take $\ddot{\varphi} = \ddot{\theta} = \dot{\varphi} = \dot{\theta} = 0$.

In our case fixpoints satisfy the equations:

$$
\begin{aligned}
0 &= \cos\theta\sin\varphi\sin\alpha - \sin\theta\cos\alpha \\
0 &= \frac{\sin\theta\cos\varphi\sin\alpha}{\sin^2\theta + \beta}
\end{aligned}
$$

The solutions to these equations are the following and can be drawn:

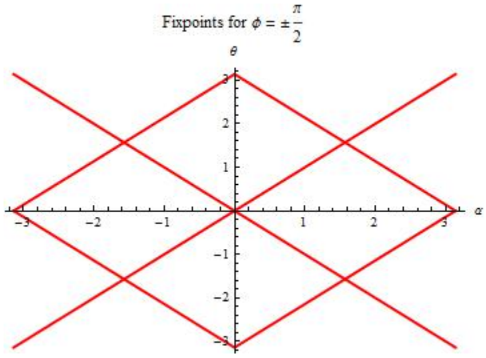


Figure 11: We find as solutions with $\varphi = \pm\frac{\pi}{2}$:

$\theta = \pm\alpha$, $\theta = \pm(\pi - \alpha)$, $\theta = \pm(\pi - \alpha)$

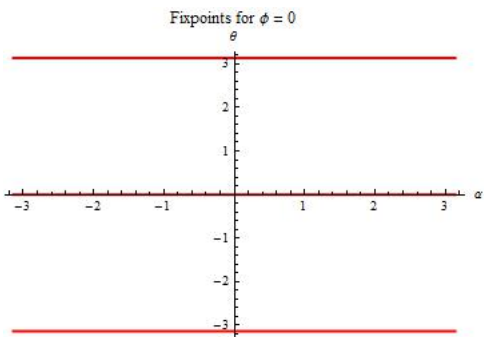


Figure 12: We find as solutions with $\varphi = 0$:

$\theta = \pm\pi$, $\theta = 0$

We will do a stability analysis of a first fixpoint which is:

$$
\begin{cases}
\theta = \alpha \\
\varphi = \frac{\pi}{2}
\end{cases}
$$

The first step is to perturb the system when it is on a fixpoint. We will consider:

$$
\begin{cases}
\theta = \alpha + \theta_1 \\
\varphi = \frac{\pi}{2} + \varphi_1
\end{cases}
\qquad \text{where } \theta_1,\varphi_1 \ll 1
$$

To determine whether a fixed point is stable or unstable, we are going to do a linear stability analysis : we place ourselves on the fixpoint, to which we add a small perturbation. The study of the perturbation's evolution will inform us on the fixpoint's stability.

We will study two fixed points.

We begin by studying the fixpoint $\theta = \alpha, \varphi = \frac{\pi}{2}$. We add a perturbation : $\theta = \alpha + \theta_1$ and $\varphi = \frac{\pi}{2} + \varphi_1$, where $\theta_1 , \varphi_1 \ll 1$ that we insert in the equations (6) and (7). After linearisation and simplification, we get :

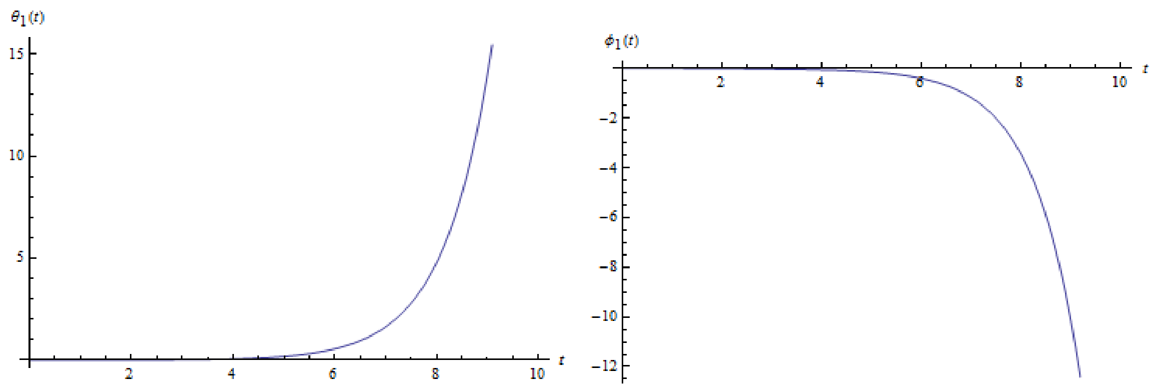
$$\begin{cases} \ddot{\theta}_1 &= -\theta_1 \\ \ddot{\varphi}_1 &= -\frac{\varphi_1}{1+\frac{\beta}{\sin^2\alpha}} \end{cases}$$

We recognise here the equations of an harmonic oscillator for each variable – thus we an deduce that the fixpoint is *stable*.

Let's now study the case of the fixpoint $\theta = \pi$, $\varphi = 0$. Proceeding as we did for the first fixpoint, we find the following equations :

$$\begin{cases} \ddot{\theta}_1 &= \theta_1 \cos\alpha - \varphi_1 \sin\alpha \\ \ddot{\varphi}_1 &= -\frac{\sin\alpha}{\beta}\theta_1 \end{cases}$$

This time we will integrate numerically these coupled equations and trace the graphs of $\theta_1(t)$ and $\varphi_1(t)$ to see the perturbations' evolution. We finally get the following curves :



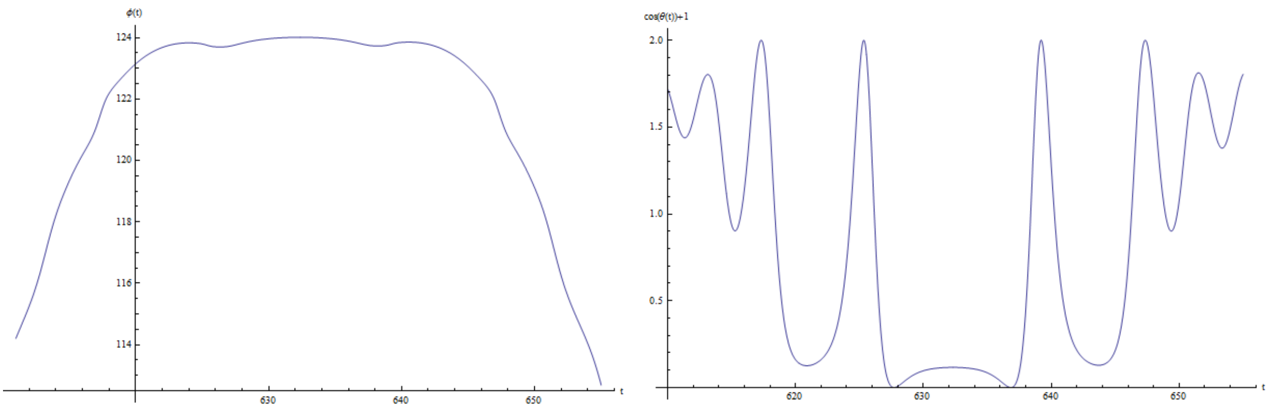
(a) Evolution of $\theta_1(t)$

(b) Evolution of $\varphi_1(t)$

Figure 13: Evolution of the perturbations

We can see that the two perturbation diverge : the considered fixpoint is therefore *unstable*.

We can observe that whenever we release the pendulum without initial speed around an unstable fixpoint, $\dot{\varphi}(t)$ sometimes changes brutally its sign – which is to say that the rotation's direction reverses. We can see that the place where the sign changes corresponds to an unstable fixpoint's position (in our case it is the fixpoint $\theta = \pi$, $\varphi = 0$).



(a) $\varphi(t)$ (b) $\cos(\theta(t)) + 1$

Figure 14: Inversion of the rotating direction around the unstable fixpoint $\theta = \pi$, $\varphi = 0$. We draw here $f(t) = \cos(\theta(t)) + 1$ to observe the points where $\theta = \pi \Leftrightarrow f(t) = 0$

To understand this behaviour, we can take once again the example of the simple pendulum : when we release it in $\theta = \pi$ without speed, it will be likely to see its rotating direction change when it passes by this point again.

VII. CONCLUSION

Beginning by the classical mechanics case that is the simple pendulum, and then adding multiple degrees of freedom and parameters, we have been able to find the movement equations of a chaotic system. We also have succeeded to visualize the its trajectories by numerical integration. The study of its Poincaré sections helped us to characterize the chaos and our system's behaviour. Moreover, nondimensionalizing the equations helped to simplify them and make the stablity analysis of the fixed points easier. We finally could find and characterise the chaotic pendulum's fixed points, and observe a critical behaviour leading to the inversion of the rotating direction.

This project allowed us to have a numerical approach of a chaotic motion, starting from a simple system. It also contributed to strengthen our knowledge in non-linear dynamics, and in making us discover some new notions and techniques (Poincaré sections, computer languages, programming).

VIII. ACKNOWLEDGEMENTS

We particularly want to thank M. Argentina and Y. Bouret, our tutors during this laboratory project.

We also thank V. Doya who organised these projects, the J.-A. Dieudonné laboratory, the Nice - Sophia Antipolis university, Wolfram Mathematica, and our classmates for their support.

[1] P. Bergé, Y. Pomeau and C. Vidal, *L'Ordre dans le chaos* (Hermann, 1988)

[2] L. Landau and E. Lifchitz, *Physique Théorique, vol. 1 : Mécanique* (MIR editions, 1982)

[3] E. Goncalves, *Introduction aux systèmes dynamiques et chaos* (April 2004) `http://cel.archives-ouvertes.fr/docs/00/55/69/72/PDF/IntroductionSystemesDynamiquesEtChaos_EricGoncalves.pdf`

[4] V. Mallet, *Contrôle de systèmes chaotiques* (2012) `http://vivienmallet.net/chaos/index.php`

[5] LASIN, Univerza v Ljubljani, *Calculation and plot of the poincare section of two uncoupled osciators* (May 2009) `http://lab.fs.uni-lj.si/lasin/www/teaching/chaotic/Poincare/poincare1.html`